Vertex-, edge-, and total-colorings of Sierpiński-like graphs

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Abstract

Vertex-colorings, edge-colorings and total-colorings of the Sierpiński gasket graphs $S_n$, the Sierpiński graphs $S(n, k)$, graphs $S^+(n, k)$, and graphs $S^{++}(n, k)$ are considered. In particular, $\chi''(S_n)$, $\chi'(S(n, k))$, $\chi(S^+(n, k))$, $\chi(S^{++}(n, k))$, $\chi'(S^+(n, k))$, and $\chi'(S^{++}(n, k))$ are determined.

Key words: Sierpiński gasket graphs; Sierpiński graphs; chromatic number; chromatic index; total chromatic number

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1 Introduction

Graphs of “Sierpiński type” appear naturally in many different areas of mathematics as well as in several other scientific fields. One of the most important families of such graphs is formed by the Sierpiński gasket graphs—the graphs obtained after a finite number of iterations that in the limit give the Sierpiński gasket, see, for instance, [8]. These graphs were introduced already in 1944 by Scorer, Grundy and Smith [20]. Among others, the Sierpiński gasket graphs play an important role in dynamic systems and probability, cf. [7, 9], as well as in psychology, cf. [14].

Sierpiński gasket graphs are just a step from the Sierpiński graphs \( S_n \). The graph \( S_n \) is obtained from \( S(n, 3) \) by contracting every edge of \( S(n, 3) \) that lies in no triangle. This connection was already observed in psychological literature by Sydow back in 1970 [21]. One of the main features of the graphs \( S(n, 3) \) is that they are precisely the graphs of the Tower of Hanoi puzzle with \( n \) discs. These graphs were quite extensively studied by now, see, for instance, [1, 4, 5, 12, 19].

In [11], the graphs \( S(n, 3) \) were generalized to the Sierpiński graphs \( S(n, k) \) for \( k \geq 3 \). The motivation for this generalization came from topological studies of the Lipscomb's space [15, 16]. (We note that the Sierpiński graphs independently appeared in [18].) As it turned out, the graphs \( S(n, k) \) possess many appealing properties, as for instance several coding [3] and several metric properties [17]. The generalization of \( S(n, 3) \) to \( S(n, k) \) is done via a certain labeling technique (see Section 2) that in turn gives a new powerful tool for studying the classical Tower of Hanoi graphs \( S(n, 3) \). The labeling technique has been fruitfully applied in [4, 19].

The graphs \( S(n, k) \) are almost regular and there are at least two natural ways to extend them to regular graphs. In this spirit regularizations \( S^+(n, k) \) and \( S^{++}(n, k) \) were proposed in [13]. For these two families of graphs the exact crossing number can be determined (modulo the crossing number of complete graphs), thus they present the first known examples of graphs of “fractal” type for which this can be done [13].

Besides the mentioned properties, vertex and edge colorings of the graphs \( S_n \) and \( S(n, k) \) were previously studied. Teguia and Godbole [22] showed that \( \chi(S_n) = 3 \). In fact, these colorings are unique [10]. In the latter paper it is also proved that for any \( n \geq 2 \), \( \chi'(S_n) = 4 \). Teguia and Godbole [22] asked what is the total chromatic number of Sierpiński gasket graphs. We answer their question in Section 3.

Parisse [17] noticed that \( \chi(S(n, k)) = k \). In Section 4 we determine the chromatic index of these graphs and the total chromatic number when \( k \) is odd. We also show that the famous Behzad-Vizing conjecture also holds when \( k \) is even.

In the last section we consider vertex-, edge-, and total-colorings of the graphs \( S^+(n, k) \) and \( S^{++}(n, k) \), and in particular determine their chromatic number and chromatic index.

The results obtained in this paper together with the previously known results are collected in Table 1.
2 Preliminaries

Let $G$ be a graph. Recall that the chromatic number $\chi(G)$ (chromatic index $\chi'(G)$) is the smallest number of colors needed for a proper vertex-coloring (edge-coloring) of $G$, where proper vertex-coloring (edge-coloring) means that adjacent vertices (edges) receive different colors. Clearly, $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ denotes the largest degree of $G$. Vizing’s theorem asserts that in addition $\chi'(G) \leq \Delta(G) + 1$. We will show that $\chi'(S_n) = \Delta(S_n) + 1$ for $n \geq 2$ and $k$ even.

The total chromatic number $\chi''(G)$ is the smallest number of colors needed for a proper coloring of both vertices and edges of $G$. Clearly, $\chi''(G) \geq \Delta(G) + 1$. Recall that $\chi''(K_n) = \Delta(K_n) + 1$ if $n$ is odd and $\chi''(K_n) = \Delta(K_n) + 2$ if $n$ is even, see [23]. Behzad-Vizing conjecture claims that $\chi''(G) \leq \Delta(G) + 2$. This conjecture has been verified for several classes of graphs, see [2, 23, 25] and references therein. All the graphs studied in this paper support the conjecture.

In the rest of this section we define the families of Sierpiński-like graphs considered in this paper.

We begin with the Sierpiński graphs $S(n, k)$ that are defined for $n \geq 1$ and $k \geq 1$.

<table>
<thead>
<tr>
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<th>$\chi$</th>
<th>$\chi'$</th>
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<tr>
<td>$S_n$</td>
<td>3 (uniquely)</td>
<td>4</td>
<td>5</td>
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<tr>
<td>$S(n, k)$</td>
<td>$k$</td>
<td>$n \geq 2, k \geq 2$</td>
<td>$k+1$</td>
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<tr>
<td>$S^+(n, k)$</td>
<td>$k$</td>
<td>$n \geq 2, k \geq 2, k$ even</td>
<td>$k+1 \leq \cdot \leq k+2$</td>
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<tr>
<td>$S^{++}(n, k)$</td>
<td>$k$</td>
<td>$n \geq 2, k \geq 2$</td>
<td>$k+1 \leq \cdot \leq k+2$</td>
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Table 1: Summary of the results
as follows. The vertex set of $S(n, k)$ consists of all $n$-tuples of integers $1, 2, \ldots, k$, that is, $V(S(n, k)) = \{1, 2, \ldots, k\}^n$. Two different vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are adjacent if and only if there exists an $h \in \{1, \ldots, n\}$ such that

(a) $u_t = v_t$, for $t = 1, \ldots, h - 1$;
(b) $u_h \neq v_h$; and
(c) $u_t = v_h$ and $v_t = u_h$ for $t = h + 1, \ldots, n$.

We will write $\langle u_1 u_2 \ldots u_n \rangle$ for $(u_1, u_2, \ldots, u_n)$ or even shorter $u_1 u_2 \ldots u_n$. See Fig. 1 for $S(3, 4)$.

![Figure 1: The Sierpiński graph $S(3, 4)$](image)

The vertices $\langle i \ldots i \rangle$, $i \in \{1, \ldots, k\}$, are called the extreme vertices of $S(n, k)$. For $i = 1, 2, \ldots, k$ let $S_i(n + 1, k)$ be the subgraph of $S(n + 1, k)$ induced by the vertices of the form $\langle i \ldots \rangle$. Clearly, $S_i(n + 1, k)$ is isomorphic to $S(n, k)$. Consequently, $S(n + 1, k)$, $k > 2$, contains $k^n$ copies of the graph $S(1, k) = K_k$. The edges of $S(n, k)$ that lie in no induced $K_k$ will be called linking edges.

The Sierpiński gasket graph $S_n$, $n \geq 1$, is obtained from $S(n, 3)$ by contracting all the edges of $S(n, 3)$ that lie in no triangle, see Fig. 2 for $S_4$.

Following [10] we label the vertices of $S_n$ as follows. Let $\langle u_1 \ldots u_r i j \ldots \rangle$ and $\langle u_1 \ldots u_r j i \ldots \rangle$ be endvertices of an edge of $S(n, 3)$ that is contracted to a vertex $x$ of $S_n$. Then label $x$ with $\langle u_1 \ldots u_r \rangle \{i, j\}$, where $r \leq n - 2$. In this way $S_n$ has
three special vertices \( \langle 1 \ldots 1 \rangle \), \( \langle 2 \ldots 2 \rangle \), and \( \langle 3 \ldots 3 \rangle \), called \textit{extreme vertices} of \( S_n \), together with the vertices of the form

\[
\langle u_1 \ldots u_r \rangle \{i, j\},
\]

where \( 0 \leq r \leq n - 2 \), and all the \( u_k \)'s, \( i \) and \( j \) are from \( \{1, 2, 3\} \). Note that \( S_{n+1} \) contains three isomorphic copies of \( S_n \), a fact utmost useful for inductive arguments. We will denote these copies with \( S_{n+1,i} \), \( 1 \leq i \leq 3 \), where \( S_{n+1,i} \) is the subgraph \( S_n \) of \( S_{n+1} \) containing \( \langle i \ldots i \rangle \).

The extended Sierpiński graphs \( S^+(n,k) \) and \( S^{++}(n,k) \) were introduced in [13] in the following way. The graph \( S^+(n,k) \), \( n \geq 1 \), \( k \geq 1 \), is obtained from \( S(n,k) \) by adding a new vertex \( w \), called the \textit{special vertex} of \( S^+(n,k) \), and edges joining \( w \) with all extreme vertices of \( S(n,k) \). These edges will be called the \textit{additional edges} of \( S^+(n,k) \). See Fig. 3 for \( S^+(3,3) \). Note that contrary to the construction of the
$S(n, k)$ graphs, the graph $S^+(n, k)$ is not composed of $k$ copies of $S^+(n - 1, k)$.

The graphs $S^{++}(n, k)$, $n \geq 1$, $k \geq 1$, are defined as follows. For $n = 1$ we set $S^{++}(1, k) = K_{k+1}$. Suppose now that $n \geq 2$. Then $S^{++}(n, k)$ is the graph obtained from the disjoint union of $k + 1$ copies of $S(n - 1, k)$ in which the extreme vertices in distinct copies of $S(n - 1, k)$ are connected as the complete graph $K_{k+1}$. With this the graphs $S^{++}(n, k)$ are well defined, see [13, Lemma 2.2]. See Fig. 3 for $S^{++}(3, 3)$.

![Graphs $S^+(3, 3)$ and $S^{++}(3, 3)$](image)

Figure 3: Graphs $S^+(3, 3)$ and $S^{++}(3, 3)$

Note that $S^{++}(n, k)$ can also be described as the graph obtained from the disjoint union of a copy of $S(n, k)$ and a copy of $S(n - 1, k)$ such that the extreme vertices of $S(n, k)$ and the extreme vertices of $S(n - 1, k)$ are connected by a matching.

3 Total colorings of $S_n$

In this section we answer a question from [22] with the following result.

**Theorem 3.1** For any $n \geq 2$, $\chi''(S_n) = 5$.

**Proof.** Since $n \geq 2$, $\chi''(S_n) \geq \Delta(S_n) + 1 = 5$, we only need to construct a total coloring with five colors. For a total coloring $c$ of $S_n$ we will use the following notation. Let $\{i, j, k\} = \{1, 2, 3\}$. Then if $c(i \ldots i) = x$, $c(i \ldots i \langle i \ldots i \{i, j\}\rangle) = y$, and $c(\langle i \ldots i \langle i \ldots i \{i, k\}\rangle\rangle) = z$, we will write $C_i = \{x, \{y, z\}\}$.

First we construct a coloring of $S_2$ with $C_1 = \{3, \{1, 2\}\}$, $C_2 = \{4, \{1, 5\}\}$ and $C_3 = \{5, \{1, 2\}\}$. These attributions imply the colors of the remaining elements, as shown in Fig. 4. Then color $S_3$ as follows. Let $c'$ be a coloring of $S_{3,1}$ such that
Let $c'$ be a coloring of $S_{3,2}$ such that $C'_1 = \{4, \{2, 3\}\}$, $C'_2 = \{1, \{3, 5\}\}$, and $C'_3 = \{2, \{3, 5\}\}$. Finally, let $c''$ be a coloring of $S_{3,3}$ with $C''_1 = \{5, \{3, 4\}\}$, $C''_2 = \{2, \{1, 4\}\}$, and $C''_3 = \{3, \{1, 4\}\}$. Note that $c' = c$, and that $c''$ and $c'''$ are obtained from $c'$ by applying permutations $(13)(25)(4)$ and $(14532)$, respectively. The coloring of $S_3$ is schematically shown on the right-hand side of Fig. 4.

Let $c$ be the constructed coloring of $S_3$, then $C_1 = \{3, \{1, 2\}\}$, $C_2 = \{1, \{3, 5\}\}$, and $C_3 = \{3, \{1, 4\}\}$. Next color $S_4$ as follows. Let $c' = c$ be a coloring of $S_{4,1}$, let $c''$ be a coloring of $S_{4,2}$ such that $C''_1 = \{3, \{4, 5\}\}$, $C''_2 = \{4, \{3, 1\}\}$, and $C''_3 = \{3, \{4, 2\}\}$, and let $c'''$ be a coloring of $S_{4,3}$ with $C'''_1 = \{3, \{2, 5\}\}$, $C'''_2 = \{3, \{1, 5\}\}$, and $C'''_3 = \{5, \{3, 4\}\}$.

For $n \geq 4$ we proceed by induction. Suppose that $c$ is a total coloring of $S_n$ with $C_1 = \{1, \{3, 5\}\}$, $C_2 = \{4, \{1, 3\}\}$, and $C_3 = \{5, \{3, 4\}\}$. Then let $c' = c$ be a coloring of $S_{n+1,1}$, let $c''$ be a coloring of $S_{n+1,2}$ with $C''_1 = \{4, \{2, 5\}\}$, $C''_2 = \{5, \{2, 3\}\}$, and $C''_3 = \{3, \{2, 4\}\}$. Finally, let $c'''$ be a coloring of $S_{n+1,3}$ with $C'''_1 = \{5, \{1, 2\}\}$, $C'''_2 = \{3, \{1, 5\}\}$, and $C'''_3 = \{2, \{1, 3\}\}$. Colorings $c', c''$, and $c'''$ exist by induction. (Note that $c''$ and $c'''$ are obtained from $c'$ using permutations of colors $(1425)(3)$ and $(154)(2)(3)$, respectively.) See the right-hand side of Fig. 5.

Now, $S_{n+1}$ is colored with a coloring $c$ where $C_1 = \{1, \{3, 5\}\}$, $C_2 = \{5, \{2, 3\}\}$, and $C_3 = \{2, \{1, 3\}\}$. Finally, exchange the role of colors 2 and 4 in $c$ and apply the induction.

4 Edge- and total-colorings of $S(n,k)$

In [10] it is shown that $S(n,3)$ is uniquely 3-edge colorable. In this section we first extend this result by proving that for any $k$, $\chi'(S(n,k)) = k$. 

7
Theorem 4.1 For any \( n \geq 2 \) and any \( k \geq 2 \), \( \chi'(S(n,k)) = k \).

Proof. If \( k \) is even the conclusion is easy. Each subgraph \( K_k \) of \( S(n,k) \) can be edge-colored with \( k-1 \) colors. Color the remaining edges, that is, the linking edges of \( S(n,k) \), with color \( k \) to obtain a desired coloring of \( S(n,k) \).

Let now \( k \) be odd. For a vertex \( u \) of \( S(n,k) \) and an edge-coloring \( c \) of it we will write \( C_u \) to denote the set of colors assigned to the edges incident with \( u \). We will prove the following stronger claim.

Claim: For any \( n \geq 1 \) and any \( k \geq 2 \), \( \chi'(S(n,k)) = k \). Moreover, for any \( i, j \in \{1, \ldots, k\} \), \( i \neq j \), \( C_{i \ldots i} \neq C_{j \ldots j} \).

For \( n = 1 \), \( S(1,k) = K_k \). It is well-known that \( K_k \) can be edge-colored with \( k \) colors such that \( C_i \neq C_j \) for \( i \neq j \). Assume the claim holds for \( n \geq 1 \). We wish to find an edge-coloring of \( S(n+1,k) \). By the induction assumption, \( S_h(n+1,k) \), \( h \in \{1, \ldots, k\} \), can be colored with \( k \) colors where \( C_{hi \ldots i} \neq C_{hj \ldots j} \), \( i, j \in \{1, \ldots, k\} \), \( i \neq j \). Let \( M \) be a mapping

\[
M : \{ij \ldots j \in V(S(n,k)) \mid i, j \in \{1, \ldots, k\}\} \to \{0, 1, \ldots, k-1\}
\]

defined as

\[
M(ij \ldots j) = i + j - 2 \pmod{k}.
\]

Let \( u = ij \ldots j \) and \( v = il \ldots l \) be two different extreme vertices of \( S_i(n,k) \). Then \( M(ij \ldots j) = i + j - 2 \pmod{k} = i + l - 2 \pmod{k} = M(il \ldots l) \), because \( i \) is fixed and \( j \neq l \). Since \( S_i(n,k) \) is isomorphic to \( S(n-1,k) \), by the induction assumption \( \chi'(S_i(n,k)) = k \) and for any two different extreme vertices \( ij \ldots j \) and \( il \ldots l \), \( C_{ij \ldots j} \neq C_{il \ldots l} \). The mapping \( M \) also assigns pairwise different numbers of the set \( \{0, \ldots, k-1\} \) to the extreme vertices. Permute the colors of the proper edge-coloring of the graph \( S_i(n,k) \) in such a way that \( C_{ij \ldots j} = \{0, \ldots, k-1\} \setminus \{M(ij \ldots j)\} \).
Consider the edges that connect subgraphs $S_i(n,k)$ and $S_j(n,k)$, for any $i, j \in \{1, \ldots, k\}$, $i \neq j$. Since $M(ij \ldots j) = M(ji \ldots i)$, the same color is missing at $ij \ldots j$ and $ji \ldots i$. Hence the edge between these two vertices can be colored with $M(ij \ldots j)$ and we have constructed a proper $k$-edge-coloring of $S(n,k)$.

To complete the proof we need to prove that the extreme vertices receive pairwise different colors. For an extreme vertex $ii \ldots i$ we have

$$M(ii \ldots i) = i + i - 2 \pmod{k} = 2(i - 1) \pmod{k}.$$ 

Recall that $k$ is odd. Hence, if $2(i - 1) < k$, the extreme vertices receive pairwise different even numbers, while if $2(i - 1) > k$, they receive pairwise different odd numbers. Finally, replace color 0 with $k$. □

In the rest of this section we consider total colorings of Sierpiński graphs. We first observe:

**Proposition 4.2** For any $n \geq 1$ and any $k \geq 1$, $\chi''(S(n,k)) \leq k + 2$.

**Proof.** Totally color every induced $K_k$ of $S(n,k)$ with at most $k+1$ colors. Moreover, color them identically, that is, two vertices with the same last coordinate from different copies of $K_k$ receive the same color. Hence any linking edge connects vertices of different colors. At the end color the linking edges with $k + 2$. □

If $k$ is odd, it is not difficult to give the exact value of the total chromatic number.

**Proposition 4.3** For any $n \geq 2$ and any odd $k \geq 3$, $\chi''(S(n,k)) = k + 1$.

**Proof.** As in the previous proof color identically every induced $K_k$ of $S(n,k)$ with $k$ colors and color the remaining edges with $k + 1$. □

When $k$ is even, the situation is more involved. Note first that $S(n,2)$ is the path on $2^n$ vertices, hence $\chi''(S(n,2)) = 3$. Next, for $k = 4$ we have:

**Proposition 4.4** For any $n \geq 1$, $\chi''(S(n,4)) = 5$.

**Proof.** Let $n \geq 2$ and let $c$ be a total coloring of $S(n,4)$. For any $i, j \in \{1, 2, 3, 4\}$ set $C_{ij \ldots j} = (a, b)$, where $c(ij \ldots j) = a$ and $b$ is a color that is neither assigned to $ij \ldots j$ nor any of its incident edges. Note that $b$ is uniquely determined since $k = 4$. For $n = 1$ set $C_1 = (4,1)$, $C_2 = (1,2)$, $C_3 = (2,3)$, and $C_4 = (3,4)$, see Fig. 6.

The result will follow from the following stronger claim.

**Claim:** If $n$ is odd, then we can color $S(n,4)$ such that $C_{i1 \ldots 1} = (4,1)$, $C_{i2 \ldots 2} = (1,2)$, $C_{i3 \ldots 3} = (2,3)$, and $C_{i4 \ldots 4} = (3,4)$. If $n$ is even, we can color $S(n,4)$ such that $C_{i1 \ldots 1} = (4,1)$, $C_{i2 \ldots 2} = (4,3)$, $C_{i3 \ldots 3} = (4,5)$, and $C_{i4 \ldots 4} = (4,2)$.
Figure 6: Cases $n = 1$ and $n = 2$

Note that we can exchange the values of $C_{i_2...2}$ and $C_{i_4...4}$, if we mirror the coloring with respect to the diagonal between vertices $i_1...i_1$ and $i_4...i_4$. Let us call the coloring from the claim the standard coloring and the derived one the mirror coloring of $S(n, 4)$.

For $n = 1$ and $n = 2$ the claim holds by Fig. 6.

Let $n \geq 2$ be even. We will construct four different total colorings $c_i$, $1 \leq i \leq 4$, of $S(n, 4)$ and combine them to a total coloring of $S(n + 1, 4)$.

Let $c_1$ be the standard coloring of $S(n, 4)$ such that $C_{11...1} = (4, 1)$, $C_{12...2} = (4, 3)$, $C_{13...3} = (4, 5)$ and $C_{14...4} = (4, 2)$. Let $c_2$ be the standard coloring such that $C_{21...1} = (2, 3)$, $C_{22...2} = (2, 4)$, $C_{23...3} = (2, 1)$ and $C_{24...4} = (2, 5)$. Note that $c_2$ is obtained from $c_1$ by the permutation $(1\, 3\, 4\, 5)$. Applying the permutation $(1\, 5\, 2\, 4\, 3)$ to $c_1$ we obtain the standard coloring $c_3$ for which $C_{31...1} = (3, 5)$, $C_{32...2} = (3, 1)$, $C_{33...3} = (3, 2)$ and $C_{34...4} = (3, 4)$. Finally, using $(1\, 2\, 3\, 5\, 4)$ the standard coloring $c_4$ is obtained for which $C_{41...1} = (1, 2)$, $C_{42...2} = (1, 5)$, $C_{43...3} = (1, 4)$, and $C_{44...4} = (1, 3)$. Now color $S(n + 1, 4)$ in such a way that $C_{11...1} = (4, 1)$, $C_{22...2} = (2, 4)$, $C_{33...3} = (3, 2)$, and $C_{44...4} = (1, 3)$.

Combine the colorings $c_i$ to a coloring of $S(n + 1, 4)$ as shown in the left-hand side of Fig. 7. From this it is clear that the linking edges of $S(n + 1, k)$ can be properly colored (with the missing colors between the corresponding vertices). To complete the even to odd case apply the mirror coloring to get $C_{11...1} = (4, 1)$, $C_{22...2} = (1, 3)$, $C_{33...3} = (3, 2)$, and $C_{44...4} = (2, 4)$. Finally, the exchange of colors 2 and 3 yields the desired coloring of $S(n + 1, 4)$, where $n + 1$ is odd.

Let $n \geq 3$ be odd. As in the previous case we first construct four different total colorings $c_i$, $1 \leq i \leq 4$, of $S(n, 4)$. Let $c_1$ be the standard coloring such that $C_{11...1} = (4, 1)$, $C_{12...2} = (1, 2)$, $C_{13...3} = (2, 3)$, and $C_{14...4} = (3, 4)$. Let $c_2$ be the mirror coloring of the coloring obtained from $c_1$ by permuting the colors as $(1\, 2\, 5)(3\, 4)$. In this case, $C_{21...1} = (3, 2)$, $C_{22...2} = (4, 3)$, $C_{23...3} = (5, 4)$, and
Figure 7: Even to odd and odd to even cases

$C_{24,4} = (2,5)$. Let $c_3$ be the standard coloring obtained from $c_1$ by means of $(13524)$. Then $C_{31..1} = (1,3)$, $C_{32..2} = (3,4)$, $C_{33..3} = (4,5)$, and $C_{34..4} = (5,1)$. The last coloring, $c_4$, is the mirror coloring of the coloring obtained from $c_1$ using $(1453)(2)$. Then $C_{41..1} = (5,4)$, $C_{42..2} = (1,5)$, $C_{43..3} = (2,1)$ and $C_{44..4} = (4,2)$. Now combine $c_1$, $c_2$, $c_3$, and $c_4$ into $S(n + 1,4)$ as shown in the right-hand side of Fig. 7. Color every linking edge with the missing color to obtain the desired total coloring.

For even $k \geq 6$ we were not able to decide whether $\chi''(S(n,k)) = k + 1$ or $\chi''(S(n,k)) = k + 2$. We do, however, suspect the following.

**Conjecture 4.5** For any even $k \geq 6$, $\chi''(S(n,k)) = k + 2$.

## 5 Colorings of $S^+(n,k)$ and $S^{++}(n,k)$

In this final section we consider the three types of colorings on the extended Sierpiński graphs $S^+(n,k)$ and $S^{++}(n,k)$.

We begin with the chromatic number for which the following natural coloring of $S(n,k)$ will be useful. Set $c((u_1 \ldots u_n)) = u_n$ for any vertex $\langle u_1 \ldots u_n \rangle$ of $S(n,k)$ to obtain a $k$-vertex-coloring of $S(n,k)$ [17]. We call this coloring the **canonical vertex-coloring** of $S(n,k)$.

Note that $S^+(n,2)$ is an odd cycle while $S^{++}(n,2)$ is an even cycle. For $k \geq 3$ we have:

**Proposition 5.1** For any $n \geq 2$ and any $k \geq 3$, 

$$\chi(S^+(n,k)) = \chi(S^{++}(n,k)) = k.$$
Proof. Let $c$ be the canonical vertex-coloring of $S(n, k)$. Recall that $V(S^+(n,k)) = V(S(n,k)) \cup \{w\}$ and color the vertices of $S^+(n,k)$ as follows:

$$c'(u) = \begin{cases} 
1; & u = k\ldots kkk, \\
2; & u = k\ldots k1k; \\
k; & u \in \{w, k\ldots k11, k1\ldots k12\}; \\
c(u); & \text{otherwise}.
\end{cases}$$

Since $k \geq 3$ it is straightforward to verify that $c'$ is a proper coloring of $V(S^+(n,k))$.

Recall that $S^{++}(n,k)$ consists of $k+1$ copies of $S(n - 1,k)$. Color $S(n,k)$ using the canonical vertex-coloring $c$. Let $c''$ be a coloring of the additional copy of $S(n - 1,k)$ defined with

$$c''(u_1\ldots u_{n-1}) = \begin{cases} 
1; & u_{n-1} = k, \\
u_{n-1} + 1; & \text{otherwise}.
\end{cases}$$

Clearly, $c''$ is a proper $k$-coloring of $S(n - 1,k)$. Since the corresponding extreme vertices of $S(n,k)$ and $S(n - 1,k)$ are assigned different colors, $c$ and $c''$ can be combined to a proper $k$-coloring of $S^{++}(n,k)$.

\[\Box\]

Proposition 5.2 For any $n \geq 2$ and any $k \geq 2$,

$$\chi'(S^+(n,k)) = \begin{cases} 
k; & k \text{ is odd}, \\
k + 1; & k \text{ is even}.
\end{cases}$$

Proof. Recall from the proof of Theorem 4.1 that when $k$ is odd, there exists a $k$-edge-coloring of $S(n,k)$ such that $C_{i\ldots i} \neq C_{j\ldots j}$, $i, j \in \{1,\ldots,k\}$, $i \neq j$, where $C_{i\ldots i}$ is the set of colors assigned to the edges incident to the vertex $i\ldots i$. Color each edge connecting the special vertex $w$ with the vertex $i\ldots i$, $i \in \{1,\ldots,k\}$, with the color of the set $\{1,\ldots,k\}\setminus C_{i\ldots i}$. Hence $\chi'(S^+(n,k)) = k$ for odd $k$.

Let $k$ be even. Graph $S^+(n,k)$ is $k$-regular, has $k^n + 1$ vertices and $k(k^n + 1)/2$ edges. Therefore $S^+(n,k)$ is an overfull graph and hence $\chi'(S^+(n,k)) = k + 1$. \[\Box\]

Proposition 5.3 For any $n \geq 2$ and any $k \geq 2$, $\chi'(S^{++}(n,k)) = k$.

Proof. If $k$ is even, color each induced $K_k$ of $S^{++}(n,k)$ with $k-1$ colors and use color $k$ on the remaining edges.
Recall again from the proof of Theorem 4.1 that for $k$ odd, there exists a $k$-edge-coloring of $S(n, k)$ such that $C_{i_i...i} \neq C_{j_j...j}, \ i, j \in \{1, \ldots, k\}, \ i \neq j$. Apply the same theorem to color $S(n - 1, k)$. Using the theorem twice, the corresponding extreme vertices miss the same color. Color the edges connecting $S(n, k)$ and $S(n - 1, k)$ with the missing color to acquire a proper edge-coloring of $S^{++}(n, k)$.

It remains to consider the total-colorings.

**Proposition 5.4** For any $n \geq 2$ and $k \geq 2$, $\chi''(S^+(n, k)) \leq k + 2$.

**Proof.** Let $k$ be odd. First totally color each induced $K_k$ of $S^+(n, k)$ with $k$ colors such that each extreme vertex receives a different color. Color the linking edges with $k + 1$. Next color the additional edges in $S^+(n, k)$ with the color of the extreme vertex to which the additional edge is adjacent and replace the extreme vertex’s color with the color $k + 1$. Finally, color the special vertex of $S^+(n, k)$ with $k + 2$.

When $k$ is even, we can totally color each complete graph in $S(n, k)$ with $k + 1$ colors in such a way that $C_{i_i...i} \neq C_{j_j...j}, \ i, j \in \{1, \ldots, k\}, \ i \neq j$. Color the additional edges incident with $i...i, \ i \in \{1, \ldots, k\}$, with this missing color. Finally, color the linking edges and the remaining special vertex with $k + 2$.

**Proposition 5.5** For any $n \geq 2$ and $k \geq 2$, $\chi''(S^{++}(n, k)) \leq k + 2$.

**Proof.** Totally color each complete subgraph $K_k$ of $S^{++}(n, k)$ with at most $k + 1$ colors and use color $k + 2$ on the remaining edges.

**Proposition 5.6** For any $n \geq 2$ and any odd $k \geq 3$, $\chi''(S^{++}(n, k)) = k + 1$.

**Proof.** Totally color complete subgraphs $K_k$ of $S^{++}(n, k)$ with $k$ colors and color the linking edges and the additional edges with $k + 1$.

## 6 Concluding remarks

Theorem 4.1 has been independently obtained by Hinz and Parisse [6]. In the same paper they also determine the chromatic index of the general Tower of Hanoi graphs, that is, the graphs of the Tower of Hanoi puzzle where more than 3 pegs are allowed. Surprisingly, it turned out that the difficult case to treat was when there are fewer discs than pegs in the corresponding Tower of Hanoi problem.

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References


